



Does the Addition of Linear Damping Always Cause Instability in a Gyroscopically Stabilized System?

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This paper answers the question raised in the title of this paper and shows that it is not true for systems in which the damping matrix is indefinite. It introduces a new paradigm in the theory of linear stability that gyroscopically stabilized unstable potential systems *can* be made stable, and even exponentially stable, by the addition of linear damping. Conceptually, the paper also points to a practical methodology for adding damping to such gyroscopically stabilized potential systems to render them exponentially stable. The methodology involves the *simultaneous* use of both dissipative damping or negative velocity feedback, *and* positive velocity feedback. The methodology is illustrated in detail on two-degree-of-freedom gyroscopically stabilized potential systems. In-depth stability analysis of such systems is provided. It is shown that they can *always* be made *exponentially* stable by using an uncountably infinite number of appropriate indefinite damping matrices. A connected region is proved to exist in the space of indefinite damping matrices for which such damped gyroscopically stabilized systems are guaranteed to be exponentially stable, and this region of exponential stability is analytically delineated. Numerical studies are provided to corroborate the analytical results.

Nomenclature

a_1, a_2, a_3, a_4	= coefficients of characteristic polynomial
$\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4$	= coefficients of characteristic polynomial
b_1, b_2, c_1	= elements of Routh table
\bar{D}, \underline{D}	= diagonalized real constant damping matrix
\bar{D}, \hat{D}	= real constant symmetric damping matrix
\underline{d}	= diagonal elements of matrix \underline{D}
\bar{d}	= diagonal elements of matrix \bar{D}
$\bar{G}, \hat{G}, \underline{G}, \underline{G}$	= real skew symmetric constant gyroscopic matrix
g	= nonzero element of matrix \underline{G}
\bar{g}	= nonzero element of matrix \bar{G}
\bar{h}, \underline{h}	= expressions in Routh condition
$\bar{K}, \hat{K}, \underline{K}$	= real symmetric constant potential matrix
\bar{K}_1, \underline{K}	= positive definite constant real matrix
$\underline{k}, \underline{s}$	= elements of matrix \underline{K}_1
$\bar{k}_1, \bar{k}_2, \bar{k}_3$	= elements of matrix \bar{K}_1
\underline{M}	= positive definite constant real mass matrix
$p(\lambda)$	= characteristic polynomial
q	= column vector
t	= time, scaled time
u	= slope of ray starting from the origin
x, y	= column vectors
α	= diagonal elements of matrix \underline{D}
$\bar{\alpha}$	= diagonal elements of matrix \bar{D}
α_0	= $\sqrt{g^2 - k - 1}$
Δ	= $Trace(D)$
$\delta, \bar{\delta}$	= arbitrary real nonzero number
$\gamma, \hat{\gamma}$	= parameter describing slope of ray starting from the origin
λ	= variable in characteristic polynomial
τ	= scaled time
ζ	= $\gamma - 1$

I. Introduction

THE genesis of the question raised in the title of this paper goes back to the published work of Thompson and Tait in 1867, whose results were proved by Chetaev in the 1950s. The effort, which was begun by Tait in 1861 and which culminated in the mathematical proof provided by Chetaev, spans about 100 years. The accepted view that has been passed down to the scientific community regarding the question is that even the minutest linear dissipative damping when introduced in an unstable potential system that is gyroscopically stabilized makes the damped system unstable. This is the celebrated Kelvin–Tait–Chetaev (KTC) result, which is perhaps one of the best-known paradigms in linear stability theory, and one of its cornerstones. Though somewhat nonintuitive, it is of great practical importance in science and engineering because it correctly predicts the behavior of gyroscopically stabilized systems in which the damping matrix is positive definite [1–3].

The literature dealing with linear potential systems with indefinite damping forces in the presence of gyroscopic forces is scant. Merkin [4] considers an example in which a mono-rail car is stabilized by using gyroscopic forces. Explaining that the Kelvin–Tait–Chetaev (KTC) result would cause such a gyroscopically stabilized rail car to become unstable in its motion when subjected to dissipative damping, he uses indefinite damping. He shows that, by increasing the gyroscopic stabilization force, stability could be possible if certain conditions are satisfied. References [5,6], which deal with indefinite damping, look at various linear systems from the viewpoint of PT-symmetry breaking. Reference [5] includes a section on the influence of small damping and nonconservative positional forces on the stability of gyroscopic systems. Reference [6] includes a discussion on the stability of a system in which the damping matrix and the nonconservative force matrix in a linear system are interrelated. Reference [7] shows that gyroscopically stabilized systems can be destabilized by the addition of both positive definite and indefinite damping matrices. Some recent results on the stability of nonconservative systems are given in Ref. [8]. References [9,10] deal with positive semidefinite damping matrices.

This paper aims at the development of an understanding of the dynamical behavior of gyroscopically stabilized potential systems when the damping matrix is indefinite. Potential (conservative) systems are considered and assumed to be unstable. Furthermore, they are assumed to be suitably stabilized by using gyroscopic forces. The central result of the paper is that, by using appropriate damping forces that lead to indefinite damping matrices, damped gyroscopically stabilized systems could be stabilized, and explicit zones of exponential stability determined. To conceptually prove this

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general result, which is in contradistinction to the KTC paradigm, two-degree-of-freedom systems are considered. An important consequence is the establishment of the paradigm that an exponentially stable control methodology *can* be found for stabilizing such gyroscopically stabilized dynamical systems.

Though this paper deals with linear systems, the results obtained herein are also useful and important for understanding the stability of nonlinear systems. This is because the behavior of many nonlinear systems close to their hyperbolic equilibrium points can be linearized, and according to the Hartman-Grobman theorem [11] their behavior is topologically equivalent to that of the linearized system.

Consider the unstable potential system described by the equation

$$\tilde{M} \ddot{q} + \tilde{K} q = 0, \quad \tilde{K} \neq 0 \quad (1)$$

where q is a $2n$ -vector ($2n$ by 1 column vector), \tilde{M} is a constant real positive definite $2n$ -by- $2n$ matrix, and \tilde{K} is a real constant nonzero symmetric matrix. The dots indicate differentiation with respect to time t . Assume that this system is stabilized by the addition of a suitable gyroscopic force represented by the $2n$ -by- $2n$ real constant skew symmetric matrix \tilde{G} so that the system

$$\tilde{M} \ddot{q} + \tilde{G} \dot{q} + \tilde{K} q = 0 \quad (2)$$

is stable (see Appendix). As we shall see later, not all unstable potential systems described by Eq. (1) can be so stabilized. The addition of linear damping to this gyroscopically stabilized system yields the system described by

$$\tilde{M} \ddot{q} + (\tilde{D} + \tilde{G}) \dot{q} + \tilde{K} q = 0 \quad (3)$$

where \tilde{D} is a real constant symmetric matrix.

Using the transformation $q(t) = \tilde{M}^{-1/2} y(t)$ and premultiplying Eq. (2) by $\tilde{M}^{-1/2}$ this equation can be restated as

$$\ddot{y} + \hat{G} \dot{y} + \hat{K} y = 0 \quad (4)$$

where the $2n$ -by- $2n$ skew-symmetric matrix $\hat{G} = \tilde{M}^{-1/2} \tilde{G} \tilde{M}^{-1/2}$, and the symmetric matrix $\hat{K} = \tilde{M}^{-1/2} \tilde{K} \tilde{M}^{-1/2}$. As before, the undamped gyroscopic system described by Eq. (4) is stable. By adding a linear damping force to the system, which is described in Eq. (4), we obtain the equation of motion of a damped gyroscopically stabilized system as

$$\ddot{y} + (\hat{D} + \hat{G}) \dot{y} + \hat{K} y = 0 \quad (5)$$

where \hat{D} is a $2n$ -by- $2n$ symmetric matrix. In most dynamical systems the addition of a positive definite damping matrix \hat{D} increases the system's stability, since damping extracts energy from it. However, this somewhat intuitive line of thinking is false for the gyroscopically stabilized system (4), and the slightest addition of damping to the system $\hat{D} = \tilde{M}^{-1/2} \tilde{D} \tilde{M}^{-1/2}$, when the matrix \hat{D} is positive definite, causes it to become unstable. This result is the celebrated and long-established Kelvin-Tait-Chataev (KTC) result mentioned earlier [1-3].

Since the matrix \hat{D} is real and symmetric, it can be diagonalized by a real orthogonal matrix T , and by using a further transformation, $y(t) = Tx(t)$, Eq. (5) can be written as

$$\ddot{x} + (\bar{D} + \bar{G}) \dot{x} + \bar{K} x = 0 \quad (6)$$

in which the $2n$ -by- $2n$ matrix $\bar{G} = T^T \hat{G} T$ is skew-symmetric, the matrix $\bar{K} = T^T \hat{K} T$ is symmetric, and the matrix $\bar{D} = \text{diag}(\bar{d}_1, \bar{d}_2, \dots, \bar{d}_{2n})$ is a diagonal matrix.

The gyroscopically stabilized (unstable) potential system described by the following equation:

$$\ddot{x} + \bar{G} \dot{x} + \bar{K} x = 0 \quad (7)$$

[or equivalently, by Eq. (2)] will oftentimes be referred to as the "undamped gyroscopically stabilized system," for short.

II. Main Results

In the context of the notation established above, the main contribution of this paper is the investigation of the following question:

Given the matrix \tilde{K} (\tilde{K}) that describes an unstable potential system and given the matrix \tilde{G} (\tilde{G}) that gyroscopically stabilizes this unstable potential system as in Eq. (7) [respectively, Eq. (2)], do matrices \tilde{D} (\tilde{D}) exist such that the system described by Eq. (6) [respectively, Eq. (3)] is stable? Asymptotically (exponentially) stable? If such matrices \tilde{D} (\tilde{D}) do exist, how can they be explicitly found? Can the region of stability in the "space" of such damping matrices be exactly and analytically delineated?

The Kelvin-Tait-Chataev (KTC) paradigm assures us that this is not possible when \tilde{D} (\tilde{D}) is positive definite. Indefinite damping matrices are therefore considered in this paper, and it is shown that such matrices do not necessarily destabilize an undamped gyroscopically stabilized system. In fact, the appropriate addition of such indefinite damping matrices can make such undamped gyroscopically stabilized systems *exponentially* stable.

To prove the central idea behind the above assertion it is sufficient to consider a two-degree-of-freedom system described by Eq. (6). It is shown that such a gyroscopically stabilized unstable potential system can *always* be made exponentially stable through the addition of appropriate linear damping characterized by an indefinite damping matrix. Conceptually, this leads to the creation of a new control methodology that *simultaneously* uses *both* negative and positive velocity feedback. The cooperative interaction of these feedbacks is shown to make the damped system exponentially stable.

A two-degree-of-freedom dynamical system described by Eq. (6) with $n = 1$ can be written as

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{pmatrix} \bar{d} & 0 \\ 0 & -\bar{\alpha} \end{pmatrix}}_{\bar{D}} + \underbrace{\begin{pmatrix} 0 & \bar{g} \\ -\bar{g} & 0 \end{pmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} \bar{k}_1 & \bar{k}_3 \\ \bar{k}_3 & \bar{k}_2 \end{bmatrix}}_{\bar{K}_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0, \quad \bar{\alpha}, \bar{d} > 0 \quad (8)$$

in which the various parameters contained in the matrices \bar{D} , \bar{G} , and $\bar{K} = -\bar{K}_1$ are shown. The matrix \bar{D} is indefinite, since we assume that $\bar{\alpha}, \bar{d} > 0$.

The undamped ($\bar{d}, \bar{\alpha} = 0$) gyroscopically stabilized system described by Eq. (8) is assumed to be (marginally) stable. In the Appendix it is shown that one of the requirements for an unstable two-degree-of-freedom potential system to be gyroscopically stabilized is that the matrix \bar{K}_1 must be positive definite. Hence, $\bar{k}_1, \bar{k}_2 > 0$.

Since $\bar{k}_2 > 0$, a "scaled" time $\tau = \sqrt{\bar{k}_2} t$ (instead of t) can be used in Eq. (8). Dividing the resulting equation by \bar{k}_2 , Eq. (8) simplifies to

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{pmatrix} d & 0 \\ 0 & -\alpha \end{pmatrix}}_D + \underbrace{\begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}}_G \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} k & s \\ s & 1 \end{bmatrix}}_{K_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0, \quad \alpha, d, k > 0 \quad (9)$$

where

$$d = \frac{\bar{d}}{\sqrt{k_2}}, \quad \alpha = \frac{\bar{\alpha}}{\sqrt{k_2}}, \quad g = \frac{\bar{g}}{\sqrt{k_2}}, \quad k = \frac{\bar{k}_1}{k_2}, \quad \text{and} \quad s = \frac{\bar{k}_3}{k_2} \quad (10)$$

and the scaled matrices $D, G,$ and K_1 are

$$D = \bar{D}/\sqrt{k_2}, \quad G = \bar{G}/\sqrt{k_2}, \quad \text{and} \quad K_1 = \bar{K}_1/k_2 \quad (11)$$

Equations (8) and (9) are alternative representations of the same damped gyroscopically stabilized dynamical system. In what follows these two equivalent representations will be used alternatively. For convenience, we will continue to use dots over the x 's to denote differentiation in Eq. (9), though now they denote derivatives taken with respect to the scaled time, $\tau,$ and not $t.$ We shall refer to the matrix $\bar{K} = -K_1$ ($K = -K_1$) as the potential matrix, the matrix \bar{G} (G) as the gyroscopic matrix, and the matrix \bar{D} (D) as the damping matrix, in Eq. (8) [Eq. (9)].

The Appendix shows that for the undamped ($d = \alpha = 0$) (unstable) potential system shown in Eq. (9) to be gyroscopically stabilized we require that

- (i) $K_1 > 0,$ and
- (ii) $g^2 > 1 + k + 2\sqrt{(k - s^2)} = :1 + k + 2\sqrt{Det(K)} + \delta^2,$
 $\forall \delta \neq 0$ (12)

The first condition places a restriction on the nature of unstable potential systems that can be gyroscopically stabilized, that is, a restriction on the matrix $K;$ the second condition provides the permissible values of g required to achieve gyroscopic stabilization, that is, a restriction on the matrix $G.$

The aim is to find damping matrices D that can be added to the undamped gyroscopically stabilized system so that the damped system is asymptotically (exponentially) stable. More specifically, the following question is addressed: when the undamped potential system $\ddot{x} - K_1x = 0$ with $K_1 > 0$ [see Eq. (9)] is gyroscopically stabilized by using a particular value of g that satisfies Eq. (12), can parameters $d, \alpha > 0$ be found such that the damped system described by Eq. (9) is made exponentially stable, *without* altering that particular value of $g?$ That is, can a *given* gyroscopically stabilized system, *always* be made *exponentially* stable by the introduction of a suitable indefinite damping matrix?

Remark 1: The diagonal damping matrix \bar{D} in Eq. (8) has the element $\bar{d},$ which can be thought of as being physically generated through (i) provision of dissipative damping to the first degree of freedom, $x_1,$ or, from a controls viewpoint, (ii) negative velocity feedback control provided to the first degree of freedom, $x_1.$ The element $-\bar{\alpha}$ in $\bar{D},$ can be thought of as being physically generated through the provision of *positive* velocity feedback control provided to the second degree of freedom, $x_2.$ A similar interpretation can be given to the elements d and $-\alpha$ of the matrix D in Eq. (9). Later, we will consider how to choose which one of the two degrees of freedom receives dissipative damping (or negative velocity feedback) and which one receives positive velocity feedback. □

The characteristic polynomial, $p(\lambda),$ of the system described in Eq. (9) is given by

$$p(\lambda) = \lambda^4 + \underbrace{(d - \alpha)}_{a_1}\lambda^3 + \underbrace{(g^2 - k - 1 - ad)}_{a_2}\lambda^2 + \underbrace{(ak - d)}_{a_3}\lambda + \underbrace{(k - s^2)}_{a_4} \quad (13)$$

and its coefficients are denoted by

$$\begin{aligned} a_1 &= d - \alpha, & a_2 &= g^2 - k - 1 - ad, \\ a_3 &= ak - d, & \text{and } a_4 &= k - s^2 = Det(K_1) \end{aligned} \quad (14)$$

Observe that the signs of g and s in Eq. (9) are immaterial, since the characteristic polynomial contains their squares. In particular, the polynomial is therefore insensitive to the sign of the element g in the gyroscopic matrix $G.$

For stability, all the coefficients in Eq. (13) must be positive, so that we must have

$$\begin{aligned} a_1 &= Trace(D) = d - \alpha > 0 \rightarrow d > \alpha > 0, \\ a_2 &= g^2 - k - 1 - ad > 0 \rightarrow d < \frac{g^2 - k - 1}{\alpha}, \\ a_3 &= ak - d > 0 \rightarrow d < ak, \quad \text{and} \\ a_4 &= Det(K) = Det(K_1) > 0 \end{aligned} \quad (15)$$

Remark 2: The condition $Trace(D) > 0$ has a physical meaning when the system described by Eq. (9) is viewed as a first order dynamical system. Equation (9) can be written in first-order form as

$$\dot{z} := \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I_2 \\ K_1 & -(D + G) \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} := Az \quad (16)$$

The condition that $Trace(D) = d - \alpha := \Delta > 0$ can now be seen as the condition $\nabla \cdot (Az) = -Trace(D) < 0.$ Consider any volume V in our four-dimensional phase space described by the coordinates $z = [x_1, x_2, \dot{x}_1, \dot{x}_2]^T.$ Its rate of change with respect to time τ equals the integral of $\nabla \cdot (Az)$ taken over the volume $V.$ Hence, the condition $Trace(D) > 0$ simply says that, in order for the system to be stable, a necessary condition is that the phase-volume $V \sim e^{-\Delta\tau}.$ All phase volumes therefore must exponentially shrink. If $\Delta = 0,$ phase volumes are conserved. Lastly, if $\Delta < 0,$ the system is unstable, as also seen from the first relation in Eq. (15). □

The Routh table [12] for this system whose characteristic polynomial, $p(\lambda),$ given in Eq. (13) can be written down as follows:

$$\begin{array}{ccc} 1 & a_2 & a_4 \\ a_1 & a_3 & 0 \\ b_1 & b_2 & 0 \\ c_1 & 0 & \\ b_2 & & \end{array}$$

The necessary and sufficient condition for asymptotic stability of the system is that all the elements in the first column be positive, and also all the coefficients in the characteristic polynomial. For linear systems [see Eq. (3)], ‘‘asymptotic’’ stability is the same as ‘‘exponential’’ stability. The latter term gives a better quantitative feel for the behavior of the system and will more often be used in the paper.

Looking down the first column of the Routh table, when $d > \alpha$ the element a_1 is positive, as is also required by the first relation in Eq. (15) (also see Remark 2). For b_1 to be positive, we require that

$$(g^2 - k - 1 - ad) - \frac{a_3}{(d - \alpha)} > 0 \quad (17)$$

For c_1 to be positive, we require that

$$h := [(g^2 - k - 1 - \alpha d)(d - \alpha) - a_3]a_3 - (d - \alpha)^2 \text{Det}(K) > 0 \quad (18)$$

which implies that

$$(g^2 - k - 1 - \alpha d) > \frac{a_3}{(d - \alpha)} + \frac{(d - \alpha)\text{Det}(K)}{a_3} \quad (19)$$

where a_3 is given in Eq. (14). Clearly, if the inequality in Eq. (18) is satisfied when $\text{Det}(K) > 0$ [see the last relation in Eq. (15)] and when $a_3 > 0$ [see the third relation in Eq. (15)], then the one in Eq. (17) is also satisfied. Similarly, when the inequality in Eq. (18) is satisfied, $g^2 - k - 1 > 0$. Lastly, $b_2 = a_4 = \text{Det}(K)$.

We then have the following result.

Result 1: The system described by Eq. (9) is exponentially stable if and only if

$$(1) \quad d > \alpha > 0 \quad (20)$$

$$(2) \quad d < \alpha k \quad (21)$$

$$(3) \quad \alpha d < g^2 - k - 1 \quad (22)$$

$$(4) \quad h := [(g^2 - k - 1 - \alpha d)(d - \alpha) - (ak - d)](ak - d) - (d - \alpha)^2 \text{Det}(K) > 0, \quad \text{and (23)} \quad (23)$$

$$(5) \quad k - s^2 = \text{Det}(K) > 0 \quad (24)$$

The dependence on the gyroscopic damping, which is characterized by the parameter g , always appears in the conditions above in the form g^2 ; hence stability is insensitive, as observed before, to the sign of parameter g that is contained in the matrix G in Eq. (9).

Alternatively, the conditions that guarantee exponential stability are:

(i) the coefficients given in Eq. (14) are each positive, i.e.,

$$a_i > 0, \quad i = 1, \dots, 4, \quad \text{and} \quad (25)$$

(ii) $h > 0$ (26)

where h is given in Eq. (23).

For a given undamped gyroscopically stabilized system, described by known values of the parameters k , s , and g in Eq. (9), h is a function of α and d , and in what follows we shall denote $h(\alpha, d)$ simply by h . □

Corollary 1: In a similar manner to the conditions given in Eqs. (25) and (26), the system described by Eq. (8) is exponentially stable if and only if

(i) each of the coefficients $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4$ of its characteristic polynomial

$$p(\lambda) = \lambda^4 + \underbrace{(\bar{d} - \bar{\alpha})}_{\bar{a}_1} \lambda^3 + \underbrace{(\bar{g}^2 - \bar{k}_1 - \bar{k}_2 - \bar{\alpha} \bar{d})}_{\bar{a}_2} \lambda^2 + \underbrace{(\bar{\alpha} \bar{k}_1 - \bar{d} \bar{k}_2)}_{\bar{a}_3} \lambda + \underbrace{\text{Det}(\bar{K}_1)}_{\bar{a}_4} \quad (27)$$

shown in Eq. (27) are positive, where $\text{Det}(\bar{K}_1) = \bar{k}_1 \bar{k}_2 - \bar{k}_3$, and

$$(ii) \quad \bar{h} := [(\bar{g}^2 - \bar{k}_1 - \bar{k}_2 - \bar{\alpha} \bar{d})(\bar{d} - \bar{\alpha}) - \bar{a}_3] \bar{a}_3 - (\bar{d} - \bar{\alpha})^2 \text{Det}(\bar{K}_1) > 0 \quad (28)$$

Proof: The proof follows along the same lines as Result 1. When $\bar{d} = d, \bar{\alpha} = \alpha, \bar{g} = g, \bar{k}_1 = k, \bar{k}_2 = 1$, and $\bar{k}_3 = s$, then Eqs. (8) and (9) becomes identical and Eqs. (27) and (28) reduce to those in Eqs. (13) and (23), respectively. Notice that the characteristic polynomial is again insensitive to the sign of \bar{g} , since it appears as \bar{g}^2 in it.

The if-and-only-if conditions for exponential stability of the damped gyroscopically stabilized system described by Eq. (8) are, correspondingly,

(i) the coefficients shown in Eq. (27) are positive, i.e., $\bar{a}_i > 0$, $i = 1, \dots, 4$, and (29)

(ii) $\bar{h} > 0$ (30)

For a given undamped gyroscopically stabilized system, described by known values of the parameters $\bar{k}_i, i = 1, \dots, 3$, and \bar{g} in Eq. (8), \bar{h} in Eq. (28) is a function of $\bar{\alpha}$ and \bar{d} , and in what follows we shall often denote $\bar{h}(\bar{\alpha}, \bar{d})$ simply by \bar{h} . □

Remark 3: We consider first the special case when $k = 1$. The relations in Eqs. (20) and (21) cannot now be simultaneously satisfied, and hence asymptotic stability is not possible. Moreover, if $d \neq \alpha$ then one of the two coefficients a_1 or a_3 of the characteristic polynomial (13) is negative, and hence the system described by Eq. (9) is unstable. □

Result 2: When $k = 1$, the system described by Eq. (9) cannot be made exponentially stable by any choice of parameters $\alpha, d > 0$. Also, when $k = 1$, and $d \neq \alpha$ the system is unstable. The case $d = \alpha$ will be taken up later. □

Remark 4: Since K_1 is positive definite, $k > 0$. Assume now that $0 < k < 1$. Again, when $d \neq \alpha$, at least one of the two coefficients a_1 or a_3 of the characteristic polynomial (13) is negative, and hence the system is unstable. The case $d = \alpha$ will be taken up later. □

From the last two remarks we have the following result.

Result 3: The system described by Eq. (9) is not exponentially stable when $0 < k \leq 1$ and $\alpha, d > 0$. Alternatively stated, the damped system described by Eq. (8) cannot be made exponentially stable if $0 < k = (\bar{k}_1/\bar{k}_2) \leq 1$, for any choice of the parameters $\bar{\alpha}, \bar{d} > 0$. We shall later come back to the case when $0 < k < 1$ (see Result 5). □

Presently, having ruled out the possibility of exponentially stabilizing the gyro-stabilized system (9) when $0 < k \leq 1$ through the use of a linear indefinite damping matrix D with $\alpha, d > 0$, we now show that when $k > 1$, the undamped gyroscopically stabilized system described by Eq. (9) can always be made exponentially stable by a suitable choice of parameters $d, \alpha > 0$. In fact we will show that when $k > 1$, there always exists a region in the (α, d) plane for which the conditions given in Result 1 are satisfied by system (9).

The conditions for exponential stability for system (9) that are provided in Result 1 can be better understood from their geometrical description. Consider the first quadrant $(\alpha, d > 0)$ of the (α, d) plane as shown in Fig. 1a. Condition (20) requires that $d > \alpha > 0$, and therefore d must be chosen to lie above the (green) line $a_1 = 0$ or $d = \alpha$. Condition (21) requires that $d < \alpha k$. When $k > 1$, the (red) line $a_3 = 0$ or $d = \alpha k$ lies above the previous (green) line and so d must be chosen to lie below this (red) line. Condition (22) requires that d be chosen to lie below the hyperbola $a_2 = 0$ or $\alpha d = g^2 - k - 1$ that is shown by the blue line. Hence, to achieve exponential stability, our choice of d and α is restricted to the sort of triangular sector AOB shown in Fig. 1a, which is enclosed between the two straight lines $a_1 = 0$ and $a_3 = 0$, and the hyperbola $a_2 = 0$. The condition that $a_4 > 0$ or $\text{Det}(K_1) > 0$ is

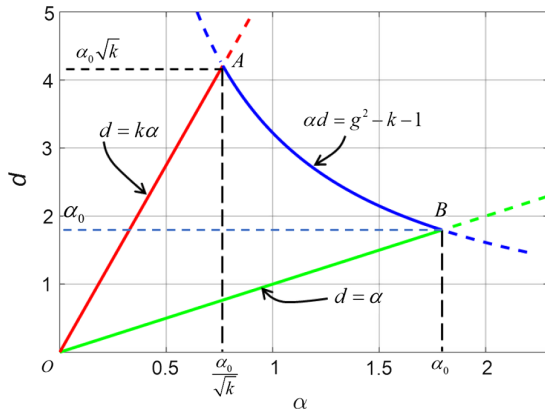


Fig. 1 a) Geometric interpretation of the Routh conditions, $\alpha_0 = \sqrt{g^2 - k - 1}$.

always satisfied since the unstable potential system is stabilizable [see Eq. (12)]. Thus the conditions for exponential stability would all be satisfied if we show that there exists a region inside the sector AOB that satisfies Eq. (23), namely, $h > 0$.

Figure 1a shows that as k is reduced from some value greater than unity, the slope of the line $d = ak$ reduces, and the region in the sector AOB between the two straight lines $d = \alpha$ and $d = ak$ reduces. These two lines appear to work like the blades of a scissor, and eventually, when $k = 1$ (or $\bar{k}_1 = \bar{k}_2$), the entire sector AOB gets completely scissored away, and the system is no longer exponentially stable. This then is the geometric insight behind Result 2. As we will see below (see Result 7), in order for the damped system to be stable when $k = 1$ (or $\bar{k}_1 = \bar{k}_2$), we would need to have $\text{Trace}(D) = 0$, and then too, only marginal stability can be achieved.

The intersection point, B , of the (green) line $d = \alpha$ and the (blue) curve $ad = g^2 - k - 1$ in the (α, d) plane has coordinates (α_0, α_0) where $\alpha_0 := \sqrt{g^2 - k - 1}$, as shown in Fig. 1a; similarly, the intersection point, A , of the (red) line $d = ak$ and the (blue) curve $ad = g^2 - k - 1$ has coordinates $(\alpha_0/\sqrt{k}, \alpha_0\sqrt{k})$.

Remark 5: For Eq. (8), the corresponding sector AOB in the $(\bar{\alpha}, \bar{d})$ plane is the sector bounded by the straight lines $\bar{a}_1 = 0$ and $\bar{a}_3 = 0$, and by the hyperbola $\bar{a}_2 = 0$ [Eq. (27)]. As before, the condition $\bar{a}_4 > 0$ is always satisfied since the unstable potential system is gyroscopically stabilizable (see the Appendix). Thus exponential stability is guaranteed if one can find a region inside this sector where $\bar{h} > 0$ [Eq. (30)]. \square

Before proving the central result, it may be useful to provide the context in which the result may be viewed. Assume that we are given a two-degree-of-freedom unstable potential system $\ddot{x} - K_1x = 0$; that is, the parameters k and s [see Eq. (9)] that describe this unstable potential system are known. This potential system is assumed to be gyroscopically stabilized. In order that this be possible, the conditions in Eq. (12) must be satisfied. Thus k and s must have values such that $K_1 > 0$, and the specific value of g chosen to stabilize the potential system must satisfy the second condition in Eq. (12) for some nonzero δ . What we want to inquire is this: Without changing the specific value of g that has been used to stabilize the unstable potential system, can parameters $d, \alpha > 0$ be so chosen in Eq. (9)—and hence the damping matrix D specified—so that the damped gyroscopically stabilized system is made exponentially stable?

Result 4: The undamped gyroscopically stabilized system described by Eq. (9) can always be made exponentially stable by a suitable choice of the parameters $\alpha, d > 0$ when $k > 1$.

Alternatively stated, consider the introduction of an indefinite damping matrix for an undamped gyroscopically stabilized system (7). The damped system described by

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{pmatrix} \bar{d} & 0 \\ 0 & -\bar{a} \end{pmatrix}}_{\bar{D}} + \underbrace{\begin{pmatrix} 0 & \bar{g} \\ -\bar{g} & 0 \end{pmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} \bar{k}_1 & \bar{k}_3 \\ \bar{k}_3 & \bar{k}_2 \end{bmatrix}}_{\bar{K}_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0, \quad \bar{k}_1 > \bar{k}_2 \quad (31)$$

can then always be made exponentially stable by a suitable choice of the parameters $\bar{\alpha}, \bar{d} > 0$.

Proof: The equation describing the system is Eq. (9) with $k > 1$. Geometrically, it must be shown that there exist points in the sector AOB in Fig. 1a for which Eqs. (20–23) are always satisfied.

From the definition of the function h given in relation (23), it is evident that the function $h < 0$ along the boundaries of the sector AOB , that is, along the lines $d = \alpha$, $d = ak$, $k > 1$, and along the hyperbola $ad = g^2 - k - 1$. Also, condition (24) is always satisfied since we are considering a gyroscopically stabilized unstable potential system, and therefore $K_1 > 0$.

Consider the line

$$d := u(\gamma)\alpha = [1 + (k - 1)/\gamma]\alpha, \quad \gamma > 1 \quad (32)$$

in the (α, d) plane (see Fig. 1a). It may be better to visualize this line as a ray that goes through the origin O with slope $u(\gamma)$, which is yet to be determined.

When $\gamma = 1$ and $u = k$, then Eq. (32) gives $d = k\alpha$, which is the line OA in Fig. 1a; when $\gamma \rightarrow \infty$ and $u = 1$, Eq. (32) gives $d = \alpha$, which is the line OB . Different values of γ yield different rays going through the origin with different slopes, and for the rays through the origin O to lie in between the two straight lines OA and OB , we must have $\gamma > 1$.

Consider a representative point P with coordinates $(\alpha, u\alpha)$ on a ray with a certain value of $\gamma > 1$. Since the undamped potential system is gyro-stabilized, g^2 must have a value that is given by Eq. (12) for some fixed value of $\delta \neq 0$. Substituting for g^2 from Eq. (12) in the expression for the function h given in Eq. (23) evaluated at our representative point P that lies on the line (32) we obtain

$$h = \frac{\alpha^2(k - 1)^2}{\gamma^3} [-\alpha^2(\gamma - 1)(\gamma + k - 1) + \gamma\{-(\gamma - 1)^2 + (\delta^2 + 2\sqrt{\text{Det}(K_1)})(\gamma - 1) - \text{Det}(K_1)\}] \quad (33)$$

This shows that when $\alpha = 0$, then $h = 0$. Our aim is to choose appropriate value(s) of $\gamma > 1$ (or slope u) when $\alpha > 0$, so that $h > 0$.

Since $\alpha > 0$ and $k, \gamma > 1$, we see from Eq. (33) that, for $h > 0$, we require the quantity in the square bracket in Eq. (33) to be positive; that is,

$$0 < \alpha^2 < \gamma \cdot \frac{\{-(\gamma - 1)^2 + [\delta^2 + 2\sqrt{\text{Det}(K_1)}]\underbrace{(\gamma - 1)}_{\zeta} - \text{Det}(K_1)\}}{(\gamma - 1)(k + \gamma - 1)} := \gamma \cdot \frac{n(\gamma)}{(\gamma - 1)(k + \gamma - 1)} \quad (34)$$

The denominator on the right-hand side of the inequality in relation (34) is always positive. We therefore need to ensure that the numerator is positive so that α is real and positive. Thinking of the quadratic $n(\gamma)$ [shown in curly brackets in the numerator in Eq. (34)] as a function of γ , values of γ for which this function is guaranteed to be positive are therefore required. For such values of $\gamma, h > 0$ when the inequality in Eq. (34) is satisfied.

Setting $\zeta := \gamma - 1 > 0$ in $n(\gamma)$, we then find that for h to be positive in the sector AOB we require ζ to be such that

$$r(\zeta) := -\zeta^2 + [\delta^2 + 2\sqrt{\text{Det}(K_1)}]\zeta - \text{Det}(K_1) > 0 \quad (35)$$

The quadratic $r(\zeta)$ is negative when $\zeta = 0$ and when $\zeta \rightarrow +\infty$, and the slope of $r(\zeta)$ is zero when $\zeta := \zeta^* = [\delta^2 + 2\sqrt{\text{Det}(K_1)}]/2 > 0$. In fact, $r(\zeta^*) = \delta^2 \sqrt{\text{Det}(K_1)} + (\delta^4/4) > 0$. Being negative when $\zeta = 0$ and $\zeta = \infty$, and positive at $\zeta = \zeta^* > 0$, the quadratic must then have two positive roots, between which it must be positive. These roots of the quadratic $r(\zeta)$ are

$$\zeta_{1,2} = \frac{1}{2}\delta^2 + \sqrt{\text{Det}(K_1)} \pm \frac{1}{2}\sqrt{\delta^4 + 4\delta^2\sqrt{\text{Det}(K_1)}} \quad (36)$$

Since $\gamma = 1 + \zeta$, for all values of $\gamma_1 < \gamma < \gamma_2$, where

$$\gamma_{1,2} = 1 + \frac{1}{2}\delta^2 + \sqrt{\text{Det}(K_1)} \mp \frac{1}{2}\sqrt{\delta^4 + 4\delta^2\sqrt{\text{Det}(K_1)}} \quad (37)$$

we therefore have $h > 0$. Thus for $h > 0$, the slope u of line (32) must lie in the open interval $(u(\gamma_1), u(\gamma_2))$.

For any given value $\hat{\gamma}$ in the (open) interval (γ_1, γ_2) , the corresponding value of the right-hand side of relation (34) gives the interval

$$0 < \alpha < \sqrt{\frac{\hat{\gamma}n(\hat{\gamma})}{(\hat{\gamma}-1)(k+\hat{\gamma}-1)}} := \alpha_{\max}(\hat{\gamma}) \quad (38)$$

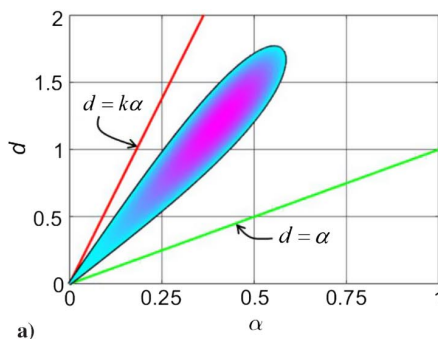
over which the function $h > 0$. For any value of $\hat{\gamma}$ so chosen, there is therefore a continuum of points along the line

$$d = [1 + (k-1)/\hat{\gamma}]\alpha := u(\hat{\gamma})\alpha, \quad 0 < \alpha < \alpha_{\max}(\hat{\gamma}) \quad (39)$$

for which $h > 0$. But h in Eq. (23) is a polynomial in α and d , and its graph is continuous. This polynomial is negative along the boundaries of the sector AOB . Along a line $d = u(\hat{\gamma})\alpha$ that lies between the lines OA and OB in Fig. 1a, $h > 0$ when α lies in the open interval $(0, \alpha_{\max}(\hat{\gamma}))$. At the two ends of this interval, $h = 0$. Because the polynomial is continuous, there must then be a region in the sector AOB around this line in which the function h continues to be positive.

Moreover, there is a continuum of such lines each described by Eq. (32) in the (α, d) plane with slopes $u(\gamma)$ for each value of γ in the (open) interval (γ_1, γ_2) . We are therefore assured that a continuous region exists in the sector AOB in which $h > 0$ and all the requirements in Result 1 are satisfied. And the existence of this region assures exponential stability of the damped gyroscopically stabilized system. \square

Remark 6: Since a considerable amount of algebra is involved in the proof, it may be useful to expose the central idea behind it. Straight lines (rays) in the (α, d) plane emanating from the origin O (see Fig. 1a) are considered. Each such ray is described by Eq. (32); it lies in the sector AOB , in between the two lines $d = \alpha$ and $d = k\alpha$, with its slope, u , controlled by the parameter $\gamma > 1$. As seen from Eq. (33), $h = 0$ when $\alpha = 0$ (i.e., at O).



For the system described by Eq. (9) to be stable it is shown that the values of γ must lie in the open interval (γ_1, γ_2) , where γ_1 and γ_2 are explicitly given in Eq. (37). For each such value of γ , say $\hat{\gamma}$, that lies in this interval, we have a line emanating from O with slope $u(\hat{\gamma}) = [1 + (k-1)/\hat{\gamma}]$ in the (α, d) plane. Starting infinitesimally close to (but excluding) O , a representative point P that lies on this line (with slope $u(\hat{\gamma})$) is allowed to travel (rightward) along it till its abscissa, α , equals $\alpha_{\max}(\hat{\gamma})$, where $\alpha_{\max}(\hat{\gamma})$ is given in Eq. (38). It is shown that $h > 0$ at every location of the point P along this line whose abscissa, α , lies in the open interval $(0, \alpha_{\max}(\hat{\gamma}))$. When $\alpha = \alpha_{\max}(\hat{\gamma})$ by Eq. (33), $h = 0$. The entire zone of stability is then determined by considering all such lines that emanate from O with their slopes u determined by values of $\hat{\gamma}$ that lie in the open interval (γ_1, γ_2) , and finding for each of them the corresponding value of $\alpha_{\max}(\hat{\gamma})$.

The proof therefore gives a constructive way of obtaining the zone of stability within the sector AOB in the (α, d) plane. (i) Use Eq. (37), to obtain the open interval (γ_1, γ_2) . For every γ in this open interval (γ_1, γ_2) draw a ray starting from O with slope $u = [1 + (k-1)/\gamma]$ that ends in the coordinate $(\alpha_{\max}, d_{\max})$ where

$$\alpha_{\max} = \sqrt{\frac{\gamma n(\gamma)}{(\gamma-1)(k+\gamma-1)}}, \quad \text{with } n(\gamma) = \{-(\gamma-1)^2 + [\delta^2 + 2\sqrt{\text{Det}(K_1)}](\gamma-1) - \text{Det}(K_1)\} \quad (40)$$

and $d_{\max} = u(\gamma)\alpha_{\max}(\gamma)$.

Then $h > 0$ at all points on every such (open) ray, excluding the ray's end points. The origin of the (α, d) plane and all such end points $(\alpha_{\max}, d_{\max})$ of each ray delineate the boundary of the region in the sector AOB where $h = 0$. The region within this boundary gives the zone of exponential stability for the system. This is how the zones of exponential stability are obtained in the examples below. \square

Remark 7: The damped gyroscopically stabilized system (9) has the characteristic polynomial given in Eq. (13); it has been shown that when $k > 1$, one can choose $\alpha, d > 0$ so that all the roots of this polynomial have negative real parts. Alternatively stated, the damped gyroscopically stabilized system (8) has the characteristic polynomial given in Eq. (27); it has been shown that when $\bar{k}_1 > \bar{k}_2$, one can choose $\bar{\alpha}, \bar{d} > 0$ so that all the roots of this polynomial have negative real parts. This remark will be used later on. \square

Remark 8: Once the zone of exponential stability is obtained in the (α, d) plane for the system described by Eq. (9), the corresponding zone of stability in the $(\bar{\alpha}, \bar{d})$ plane for the system described by Eq. (8) is simple to obtain, since from Eq. (10) we know that $\bar{\alpha} = \alpha\sqrt{k_2}$ and $\bar{d} = d\sqrt{k_2}$. \square

Hence given the undamped ($d = \alpha = 0$) gyroscopically stabilized system (9) with $k > 1$, it can always be made exponentially stable through the use of an indefinite damping matrix D ($d, \alpha > 0$), which can be provided by 1) negative velocity feedback, or dissipative damping, to the degree of freedom with coordinate x_1 , and 2) positive velocity feedback to the degree of freedom with coordinate x_2 .

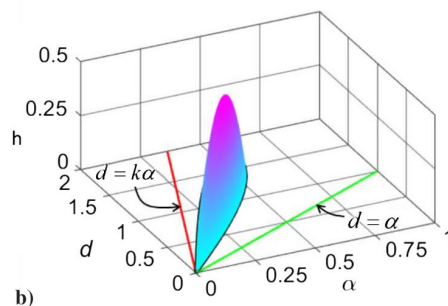


Fig. 2 a) The colored exponential stability zone. b) 3D plot of the surface h above the plane $h = 0$.

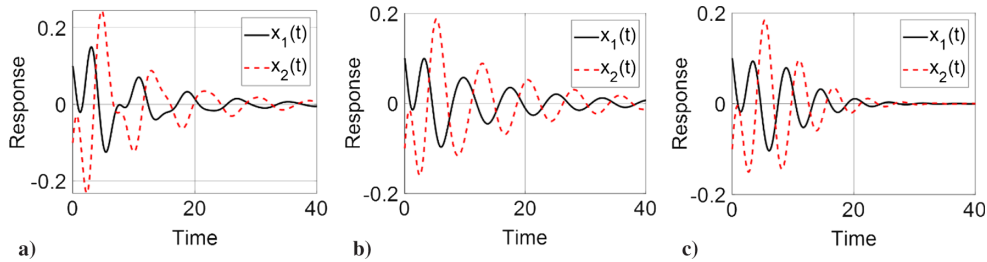


Fig. 3 Exponentially stable response: a) $\alpha = 0.25, d = 0.75$, b) $\alpha = 0.375, d = 1.25$, c) $\alpha = 0.5, d = 1.5$.

Numerical Example 1: Consider the undamped, unstable potential system described by the relation $\ddot{x} - K_1 x = 0$ with the matrix K_1 given in Eq. (9), where $k = 5.5$ and $s = 2$, so that $\text{Det}(K_1) = 1.5$. The system is gyroscopically stabilized by the matrix G using $\delta = 1$, which gives $g^2 = 9.9495$ so that Eq. (12) is satisfied. The region in the (α, d) plane in which $h > 0$ shown colored in Fig. 2a is the zone of exponential stability. The damped gyroscopically stabilized system is therefore exponentially stable for all points (α, d) that lie inside this zone. A 3D plot of the surface $h(\alpha, d) > 0$ is shown in Fig. 2b; Fig. 2a is a projection of this surface on the (α, d) plane. The hyperbola $\alpha d = g^2 - k - 1$ is outside the range of the plot and is therefore absent from it.

Responses of this damped gyroscopically stabilized systems described by Eq. (9) are illustrated in Fig. 3 for three different points (α, d) in the colored stability zone shown in Fig. 1a.

The initial conditions for the simulation (and those that follow in this paper) are: $x_1(0) = 0.1, x_2(0) = -0.1, \dot{x}_1(0) = -0.1, \dot{x}_2(0) = 0.2$.

Corollary 2: For the system described in Eq. (8), one can follow the same procedure as the proof in Result 4.

The sector AOB (see Fig. 1a) in the $(\bar{\alpha}, \bar{d})$ plane is now delineated by the two straight lines $\bar{d} = (\bar{k}_1/\bar{k}_2)\bar{\alpha} := k\bar{\alpha}$ and $\bar{d} = \bar{\alpha}$, and by the hyperbola $\bar{d}\bar{\alpha} = \bar{g}^2 - \bar{k}_1 - \bar{k}_2$. The condition that the unstable potential system can be stabilized (see the Appendix) becomes (i) $\bar{K}_1 > 0$, and (ii) $\bar{g}^2 = \bar{k}_1 + \bar{k}_2 + 2\sqrt{\text{Det}(\bar{K}_1)} + \bar{\delta}^2, \forall \bar{\delta} \neq 0$. The key stability condition now becomes $\bar{h} > 0$, where \bar{h} is given in Eq. (28).

Since the algebra becomes cumbersome, only the equations that correspond to Eqs. (32), (37), and (38), respectively, are stated below ($k = k_1/k_2$):

$$\bar{d} := u(\gamma)\bar{\alpha} = \left[1 + \frac{1}{\gamma}(k-1)\right]\bar{\alpha}, \quad \gamma > 1 \quad (41)$$

$$\gamma_{1,2} = 1 + \frac{1}{k_2} \left[\frac{1}{2}\bar{\delta}^2 + \sqrt{\text{Det}(\bar{K}_1)} \mp \frac{1}{2}\sqrt{\bar{\delta}^4 + 4\bar{\delta}^2 \text{Det}(\bar{K}_1)} \right] \quad (42)$$

and

$$0 < \bar{\alpha} < \sqrt{\frac{\hat{\gamma}n(\hat{\gamma})}{\bar{k}_2(\hat{\gamma}-1)[k+\hat{\gamma}-1]}} := \alpha_{\max}(\hat{\gamma}), \quad \text{and} \quad d_{\max} = u\alpha_{\max} \quad (43)$$

where

$$n(\hat{\gamma}) = \{-\bar{k}_2^2(\hat{\gamma}-1)^2 + \bar{k}_2[\bar{\delta}^2 + 2\sqrt{\text{Det}(\bar{K}_1)}](\hat{\gamma}-1) - \text{Det}(\bar{K}_1)\} \quad (44)$$

□

Result 4 (and Corollary 2) deals with the situation when $k > 1$ in Eq. (9) [$\bar{k}_1 > \bar{k}_2$ in Eq. (8)]. We have so far shown that under this restriction the undamped gyroscopically stabilized system is guaranteed to be exponentially stable using a set of appropriate indefinite damping matrices whose parameters lie in the stability zone, which is guaranteed to exist. Looking at the diagonal elements of matrix $K_1(\bar{K}_1)$, we observe that exponential stability of the damped gyroscopic system results when the dissipative damping

term is provided to (or “paired with”) that degree of freedom for which the diagonal element of $K_1(\bar{K}_1)$ is larger.

This, however, leaves open the question about what should be done if in an undamped gyroscopically stabilized dynamical system we do indeed have $k < 1$ in the potential matrix K_1 in Eq. (9) or, alternatively, when $\bar{k}_1 < \bar{k}_2$ in \bar{K}_1 in the undamped gyroscopically stabilized system described by Eq. (8). The next result answers this question.

Result 5: When $\bar{k}_1 < \bar{k}_2$ the undamped gyroscopically stabilized system shown in Eq. (8) with $\bar{\alpha} = \bar{d} = 0$ can always be rendered exponentially stable by *interchanging* the diagonal elements of the matrix \bar{D} in Eq. (8). That is, the system described by

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} -\bar{\alpha} & 0 \\ 0 & \bar{d} \end{bmatrix}}_{\bar{D}} + \underbrace{\begin{bmatrix} 0 & \bar{g} \\ -\bar{g} & 0 \end{bmatrix}}_{\bar{G}} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} - \underbrace{\begin{bmatrix} \bar{k}_1 & \bar{k}_3 \\ \bar{k}_3 & \bar{k}_2 \end{bmatrix}}_{\bar{K}_1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0, \quad \bar{\alpha}, \bar{d} > 0, \quad \bar{k}_1 < \bar{k}_2 \quad (45)$$

can always be made exponentially stable by an appropriate choice of $\bar{\alpha}, \bar{d} > 0$. Note in Eq. (45) that the degree of freedom, x_1 , that has the smaller value of the stiffness, \bar{k}_1 , along the diagonal of \bar{K}_1 , is provided (paired) with positive velocity feedback; the degree of freedom, x_2 , that has the larger value of the stiffness, \bar{k}_2 , along the diagonal of \bar{K}_1 is provided (paired) with dissipative damping (or negative velocity feedback). This sort of pairing is similar to that observed in Eq. (31), where we ensured exponential stability of the system when $\bar{k}_1 > \bar{k}_2$ as seen in Result 4.

Proof: Equation (45) can be rewritten as

$$\begin{bmatrix} \ddot{x}_2 \\ \ddot{x}_1 \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{d} & 0 \\ 0 & -\bar{\alpha} \end{bmatrix}}_{\bar{D}} + \underbrace{\begin{bmatrix} 0 & -\bar{g} \\ \bar{g} & 0 \end{bmatrix}}_{\bar{G}} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} - \underbrace{\begin{bmatrix} \bar{k}_2 & \bar{k}_3 \\ \bar{k}_3 & \bar{k}_1 \end{bmatrix}}_{\bar{K}_1} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 0, \quad \bar{k}_2 > \bar{k}_1 \quad (46)$$

which has exactly the same structure (matrices) as Eq. (31), except that the elements in the matrix \bar{G} now have their signs reversed. The characteristic polynomial of this equation, which is insensitive to the sign of g , is

$$\lambda^4 + (\bar{d} - \bar{\alpha})\lambda^3 + (g^2 - \bar{k}_1 - \bar{k}_2 - \bar{\alpha}\bar{d})\lambda^2 + (\bar{\alpha}\bar{k}_2 - \bar{d}\bar{k}_1)\lambda + \bar{k}_1\bar{k}_2 - \bar{k}_3^2 \quad (47)$$

Comparing Eqs. (27) and (47), we see that for the damped gyroscopically stabilized system (46), when $\bar{k}_2 > \bar{k}_1$ we can choose $\bar{\alpha}, \bar{d} > 0$ so that

all the roots of this polynomial have negative real parts (see Remark 7). Note that all the coefficients of this polynomial (47) are identical to that in Eq. (27), except for the coefficient, \bar{a}_3 , of the linear term in λ . Hence, the result. \square

Corollary 3: Result 5 is important from a practical standpoint because it informs us about *which* one of the two degrees of freedom of the undamped gyroscopically stabilized system described by Eq. (7) should be provided with dissipative damping \bar{d} so that the damped system is made exponentially stable. As seen from the matrices \bar{D} in Eqs. (31) and (45), dissipative damping (or negative velocity feedback control) should be provided to, or paired with, that degree of freedom for which the *diagonal* element of the stiffness matrix \bar{K}_1 is larger; positive velocity feedback control must be provided to, or paired with, that degree of freedom for which the *diagonal* element of the stiffness matrix \bar{K}_1 is smaller. \square

Remark 9: The meaning of Result 5 can now be further explicated. Consider the damped gyroscopically stabilized system shown in Eq. (8) in which the parameters $\bar{k}_1 = p$, $\bar{k}_2 = q$, $\bar{k}_3 = w$, and \bar{g} are given, with $p > q$. Result 4 and Corollary 2 say that such a system can always be made exponentially stable when the dissipative damping, \bar{d} , is paired with that degree of freedom for which the diagonal entry, p , of \bar{K}_1 is larger. Hence, the system

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{pmatrix} \bar{d} & 0 \\ 0 & -\bar{\alpha} \end{pmatrix}}_{\bar{D}} + \underbrace{\begin{pmatrix} 0 & \bar{g} \\ -\bar{g} & 0 \end{pmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} p & w \\ w & q \end{bmatrix}}_{\bar{K}_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0, \quad p > q \quad (48)$$

can be made exponentially stable and its zone of stability in the $(\bar{\alpha}, \bar{d})$ plane can be explicitly determined (using Remark 8 or Corollary 2). Its characteristic polynomial is [see Eq. (27)]

$$p(\lambda) = \lambda^4 + (\bar{d} - \bar{\alpha})\lambda^3 + (\bar{g}^2 - p - q - \bar{\alpha}\bar{d})\lambda^2 + (\bar{\alpha}p - \bar{d}q)\lambda + pq - w^2 \quad (49)$$

Say the stiffnesses of the diagonal elements, \bar{k}_1 and \bar{k}_2 , of \bar{K}_1 in this system are interchanged while keeping all the other parameters unchanged, so that now $\bar{k}_1 = q$, $\bar{k}_2 = p$, with, as before, $p > q$. Note that the conditions that the system with this new unstable potential matrix be gyroscopically stabilizable remain the same as before (see Appendix). Result 5 says that exponential stability is again assured when the dissipative damping, \bar{d} , is paired with the degree of freedom for which the diagonal term, p , of the new matrix \bar{K}_1 is larger. Hence the system

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{pmatrix} -\bar{\alpha} & 0 \\ 0 & \bar{d} \end{pmatrix}}_{\bar{D}} + \underbrace{\begin{pmatrix} 0 & \bar{g} \\ -\bar{g} & 0 \end{pmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} q & w \\ w & p \end{bmatrix}}_{\bar{K}_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0, \quad p > q \quad (50)$$

can be made exponentially stable and its zone of stability in the $(\bar{\alpha}, \bar{d})$ plane can be determined as proved in Result 5. The characteristic polynomial of this system, which is described by Eq. (50), is given in Eq. (47) and now becomes

$$p(\lambda) = \lambda^4 + (\bar{d} - \bar{\alpha})\lambda^3 + (\bar{g}^2 - p - q - \bar{\alpha}\bar{d})\lambda^2 + (\bar{\alpha}p - \bar{d}q)\lambda + pq - w^2 \quad (51)$$

But equations (49) and (51) are identical!

Since the systems described by Eqs. (48) and (50) have the same characteristic polynomial, all their stability properties, including their zones of exponential stability, are identical. \square

Numerical Example 2: A heavy particle rests at the origin O on a smooth surface that revolves with constant angular velocity ω about an upward vertical normal Oz , which passes through the particle. Assume that the surface has a curvature that is synclastic upward and that it is described by the equation $z = (ax_1^2 + bx_2^2)/2 + cx_1x_2$ in a coordinate frame Ox_1x_2z with origin at O that rotates with the surface. The equation of motion of the particle relative to the moving surface for small oscillations about the origin (the equilibrium position) can be written as,

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + 2 \underbrace{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} + \left(\underbrace{\begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}}_{\bar{D}} + g_e \underbrace{\begin{bmatrix} a & c \\ c & b \end{bmatrix}}_{\bar{K}_1} \right) \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0 \quad (52)$$

since z is of the second order of small quantities. Acceleration due to gravity is denoted by g_e .

Using the parameter values $a = 0.3$, $b = 0.2$, $c = 0.1$, $\omega = 2$, $g_e = 10$, in consistent units, we obtain the equation [see Eq. (45)]

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + 2 \underbrace{\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}_{\bar{K}_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0 \quad (53)$$

Since $\bar{K}_1 > 0$ and $\bar{g}^2 = 16$, we obtain $\delta^2 = 11 \neq 0$ (see the Appendix). Hence the conditions for gyroscopic stabilization are satisfied, and the unstable potential system can be stabilized when $\omega = 2$.

Since $\bar{k}_1 < \bar{k}_2$ and $\bar{k}_1 = 1$, the addition of the (indefinite) damping matrix \bar{D} to the gyroscopically stabilized system described by Eq. (53) yields the damped system

$$\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{bmatrix} -\alpha & 0 \\ 0 & d \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} + 2 \underbrace{\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}_{\bar{K}_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0, \quad \alpha, d > 0 \quad (54)$$

Equation (54) shows that, in accordance with Corollary 3, negative velocity feedback (or dissipative damping) is applied to the coordinate x_2 , and is paired with the diagonal element of the matrix \bar{K}_1 , which is larger, and whose numerical value is 2. For all values of (α, d) that lie in the zone of stability, which has been shown to always exist, this system is guaranteed to be exponentially stable. This zone of exponential stability is shown in Fig. 4a. Figure 4b shows the region in which the surface $h(\alpha, d)$ lies above the plane $h = 0$; Fig. 4a is its projection on the (α, d) plane.

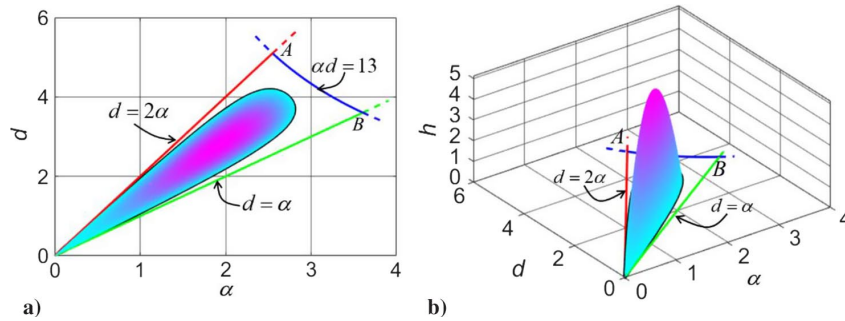


Fig. 4 a) Exponential stability zone for system in Eq. (54). b) 3D plot of the surface h above the plane $h = 0$.

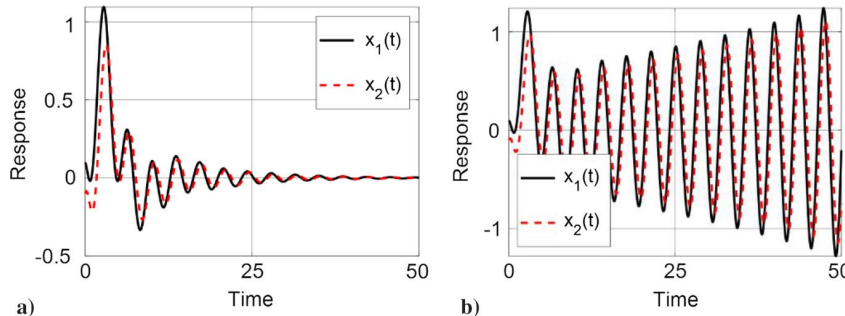


Fig. 5 a) Exponentially stable response for $\alpha = 2.7, d = 3.5$. b) Unstable response for $\alpha = 2.8, d = 3.5$.

The response of the system described by Eq. (54) for any (α, d) pair that lies in the colored region guarantees exponential stability of the damped gyroscopically stabilized system. Figures 5a and 5b show the behavior of the system for the pairs $(2.7, 3.5)$ and $(2.8, 3.5)$; the first point lies in the zone of exponential stability, whereas the second does not.

The rotating field of force described in this example is often met with in astronomy also. If the components of a binary star describe circles about their common center of mass, the force is always the same at the same point of the rotating plane. After linearization, with an appropriate change of the last matrix on the left-hand side of Eq. (52), such an equation would apply to the motion of a satellite (or planet) moving in that plane, provided that the motions of the component stars are left very nearly undisturbed by its attraction.

Some general comments can now be drawn from the nature of the zone of exponential stability illustrated in Fig. 4, some of which appear nonintuitive.

- (a) One would expect that the damped gyroscopically stabilized system would become “more stable,” were we to increase the negative velocity feedback (dissipative damping) in the system (by increasing the value of d), relative to the positive velocity feedback (α). And though we have admittedly provided this dissipative damping only to the second degree of freedom in the last example, the motion of the first degree of freedom is coupled to that of the second via the equations of motion. Because of this coupling, one might expect then that this increased stability brought on by increasing the dissipative damping on the second degree of freedom would permeate throughout the system, and make it more stable, in a sense. But it doesn’t! We see from Fig. 4a that for a given value of α for which one could have stability (say, $1 < \alpha < 2.5$), increasing the value of d and therefore the dissipative damping on the second degree of freedom does not necessarily improve the stability of the system! In fact, for values of d larger than about 4.4, this system is *always* unstable for all values of α (see Fig. 4a).
- (b) For any given value of α at which the system could be exponentially stable there is, in general, a corresponding *interval* of values of d for which the system is exponentially stable (see Fig. 4a).
- (c) While the presence of positive velocity feedback is quintessential for achieving exponential stability in the

system, its magnitude cannot be too large, since from Fig. 4 we see that the system is unstable for *all* values of α in excess of about 3.8, *no matter* what value of d is used.

- (d) As shown in Remark 9, by using Eqs. (31) and (45) with proper pairings of the elements of the damping matrix regarding the dissipative and positive feedback terms in them, the zone of exponential stability when $\bar{k}_1 < \bar{k}_2$ can be directly obtained from that for $\bar{k}_1 > \bar{k}_2$. Therefore, it is sufficient to consider only the situation when $\bar{k}_1 > \bar{k}_2$ ($k > 1$) to determine the zone of exponential stability in the (α, d) plane.

We next consider some special, nongeneric situations. When the coefficient $a_1 = 0$ in the characteristic polynomial in Eq. (9), (i) $d = \alpha$, and (ii) volumes in phase space no longer shrink, but are conserved (remain constant) as the system evolves in time (see Remark 2). We shall assume that $\alpha, d > 0$.

Remark 10: Consider the nongeneric system described by Eq. (9) when $a_1 = 0$ and $a_3 > 0$.

We note that $a_3 = d(k - 1) > 0$, when $k > 1$. The Routh table needs to be modified, since the first element in the second row of the table is 0. We then have [13]:

$$\begin{aligned} \text{Modified second row} &= (\text{original second row}) \\ &- (\text{original second row shifted to the left by one column}) \end{aligned} \quad (55)$$

Hence, we get the Routh table with the modified second row to be

$$\begin{array}{ccc} 1 & a_2 & a_4 \\ -a_3 & a_3 & 0 \\ b_1 & b_2 & 0 \\ c_1 & & \\ b_2 & & \end{array}$$

which shows that the system then is unstable. One needs only the second row of the table to deduce this.

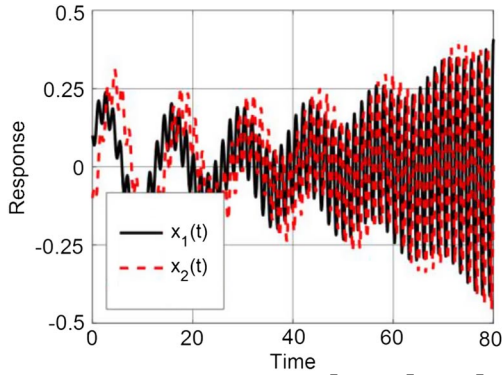


Fig. 6 Unstable response of system for $\bar{\alpha} = \bar{d} = 0.5, \bar{k}_1 = 4, \bar{k}_2 = 2$, and $\bar{g} = 5$.

When considering Eq. (8) the a_i 's in this Routh table (and in the tables that follow) are replaced by \bar{a}_i 's [see Eqs. (15) and (27)]. \square

We have then the following result.

Result 6: The undamped gyroscopically stabilized two-degree-of-freedom (nongeneric) system in Eq. (9) [Eq. (8)] is unstable when $Trace(D) = 0$ ($Trace(\bar{D}) = 0$), and $k > 1$ ($\bar{k}_1 > \bar{k}_2 > 0$). \square

Numerical Example 3: Figure 6 shows an example of the system in which $Trace(\bar{D}) = 0$, and $\bar{k}_1 > \bar{k}_2 > 0$, with the parameters

$$\bar{\alpha} = \bar{d} = 0.5, \quad \bar{k}_1 = 4, \quad \bar{k}_2 = 2, \quad \bar{k}_3 = -2, \quad \bar{g} = 5 \quad (56)$$

The response of the system is seen to be unstable.

Remark 11: When $a_1, a_3 = 0$, in the system described by Eq. (9) then every element in the second row of the Routh table is zero, and the second row needs to be modified [12]. As before, when $a_1 = 0$, then $d = \alpha$, and $Trace(D) = 0$. When $a_3 = 0$, then $d(k-1) = 0$, implying that $k = 1$ ($\bar{k}_1 = \bar{k}_2$). Recall that (see Appendix) for the unstable potential system to be gyroscopically stabilizable we now require $a_4 > 0$, and $g^2 > 2 + 2\sqrt{Det(K_1)}$. Writing the Routh table shown below

$$\begin{array}{ccc} 1 & a_2 & a_4 \\ 2 & a_2 & 0 \\ \frac{a_2}{2} & a_4 & 0 \\ a_2 - \frac{4a_4}{a_2} & & \end{array}$$

and looking down its first column we find that, if $a_2 > 0$, and $a_2^2 - 4a_4 > 0$, then there are no roots of $p(\lambda)$ in the right half complex plane. But since $p(\lambda)$ now is a polynomial that has only even powers of λ , it has no roots in the left half plane either, and hence all four roots of $p(\lambda)$ must lie on the imaginary axis. To ensure (marginal) stability, the multiplicity of each root cannot exceed unity. Using the relations in Eq. (15) for a_2 and a_4 , we have the following stability result. \square

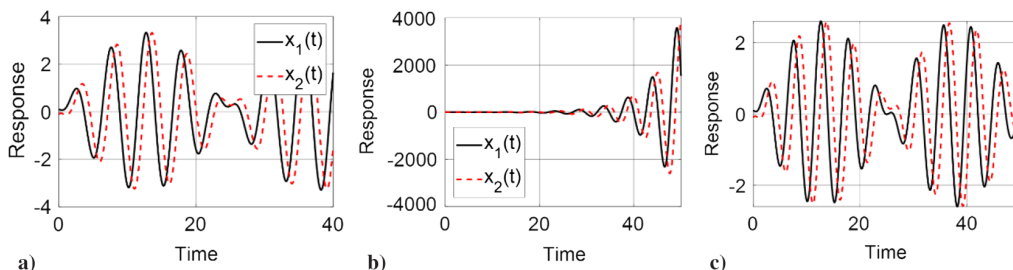


Fig. 7 $\bar{k}_1 = 2.5, \bar{k}_2 = 2.5, \bar{k}_3 = -2$: a) $\bar{\alpha} = \bar{d} = 0, \bar{g} = 2.84$. b) $\bar{\alpha} = \bar{d} = 0.8, \bar{g} = 2.92$. c) $\bar{\alpha} = \bar{d} = 0.8, \bar{g} = 2.95$.

Result 7: The two-degree-of-freedom undamped gyroscopically stabilized (nongeneric) system in Eq. (9) is marginally stable when $Trace(D) = 0$, and $k = 1$ if

$$(i) \quad g^2 > 2 + 2\sqrt{Det(K_1)} + d^2 := g_0^2, \text{ and} \quad (57)$$

$$(ii) \quad \text{The multiplicity of the imaginary roots of } p(\lambda) \text{ is unity} \quad (58)$$

Note that g_0 exceeds the minimum value of g required for gyroscopic stabilization of the undamped system.

For the system in Eq. (8), the a_i 's in the Routh table are replaced by \bar{a}_i 's [see Eqs. (15) and (27)] to get the corresponding conditions that yield marginal stability when $\bar{k}_1 = \bar{k}_2$. The first condition above then becomes

$$\bar{g}^2 > 2\bar{k}_1 + 2\sqrt{Det(\bar{K}_1)} + \bar{d}^2 := \bar{g}_0^2 \quad (59)$$

\square

Consider an unstable (nongeneric) potential system in which $\bar{k}_1 = \bar{k}_2$ that is gyroscopically stabilized. The system is marginally stable, and, say, the minimum gyroscopic force is used to stabilize it (see Appendix). If we introduce an indefinite damping matrix \bar{D} into the system, by Result 2 its $Trace(\bar{D})$ must be zero. But for the damped system to be stable, a larger gyroscopic force than that required to stabilize the undamped unstable potential system is required, as seen from relation (59). From a practical standpoint, it therefore may be unreasonable in this circumstance to add an indefinite damping matrix to such a nongeneric undamped gyroscopically stabilized system. This is because the larger gyroscopic forces needed for stability in the presence of indefinite damping still keep the system only marginally stable. The following example illustrates this.

Numerical Example 4: As an example, we take the system $\bar{\alpha} = \bar{d} = 0.8, \bar{k}_1 = 2.5, \bar{k}_2 = 2.5, \bar{k}_3 = -2, \bar{g} = 5$, in which $\bar{k}_1 = \bar{k}_2$. With these parameters, we get for the condition in Eq. (59):

$$\bar{g} > \sqrt{2\sqrt{Det(\bar{K}_1)} + 2\bar{k}_1 + \bar{d}^2} = 2.94 := \bar{g}_0$$

Hence with $\bar{g} = 5$, stability is ensured if the roots of $p(\lambda)$ have multiplicity 1, so that no polynomial-type instability occurs. The roots of the characteristic polynomial $p(\lambda)$ are $\lambda_{1,2} = \pm 0.342i$, and $\lambda_{3,4} = \pm 4.387i$, and since the multiplicity of each root is unity, the system is marginally stable.

However, the *undamped* gyroscopically stabilized system requires a gyroscopic matrix with $\bar{g} > \sqrt{2\bar{k}_1 + 2\sqrt{Det(\bar{K}_1)}} := \bar{g}_u \approx 2.83$ (see the Appendix). Since $\bar{g}_0 > \bar{g}_u$, for $\bar{g}_u < \bar{g} < \bar{g}_0$ the damped system is unstable while the undamped system is stable! Thus, on reducing the value of \bar{g} (from 5) to $2.84 > \bar{g}_u$, and thereby using a lower gyroscopic force, the unstable potential system is gyroscopically stabilized *without* any damping; its eigenvalues are $\lambda_{1,2} = \pm 1.103i$, and $\lambda_{3,4} = \pm 1.36i$, and the system is marginally stable (see Fig. 7a). Consider now increasing \bar{g} to 2.92, which is less than \bar{g}_0 yet greater than \bar{g}_u , with an indefinite damping matrix with $\bar{\alpha} = \bar{d} = 0.8$. Despite the addition of damping and the use of a greater gyroscopic

force ($\bar{g} = 2.92$), the damped system becomes unstable because Eq. (59) is no longer satisfied (see Fig. 7b). From the Routh table above, we expect to have two eigenvalues with positive real parts; their computed values are $\lambda_{1,2} = 0.169 \pm 1.213i$. The solid line shows the response $x_1(t)$ and the dashed line $x_2(t)$. Indeed, it takes an even greater gyroscopic force with $\bar{g} = 2.95 > \bar{g}_0 > \bar{g}_u$ to stabilize the system in the presence of indefinite damping as seen in Fig. 7c, while still leaving it only marginally stable with eigenvalues $\lambda_{1,2} = \pm 1.106i$, and $\lambda_{3,4} = \pm 1.356i$.

We have now covered all three cases for the unstable potential matrix \bar{K}_1 : (i) $\bar{k}_1 > \bar{k}_2$ (ii) $\bar{k}_1 < \bar{k}_2$, and (iii) $\bar{k}_1 = \bar{k}_2$. The last is a nongeneric property in the set of potential matrices. The first two cases lead to gyroscopically stabilized systems that can always be made exponentially stable by using an indefinite damping matrix whose trace is positive (Results 4 and 5, and Remark 9). The third case requires that the indefinite damping matrix have trace zero, and that the conditions in Eqs. (58) and (59) be satisfied; however, it leaves the damped gyroscopically stabilized system only marginally stability (Result 7).

Remark 12: We next consider the nongeneric case when $a_1 = 0$ and $-\beta := a_3 < 0$ in Eq. (9). These conditions imply that $0 < k < 1$. With these conditions, the Roth table becomes:

$$\begin{array}{ccc} 1 & a_2 & a_4 \\ \beta & -\beta & 0 \\ a_2 + 1 & a_4 & 0 \\ -\beta - \frac{\beta a_4}{a_2 + 1} & & \end{array}$$

To ensure that the first column of the Routh table is positive, we require that $a_2 + 1 > 0$. But then in the fourth row there is a sign change since $a_4 = \text{Det}(K) > 0$, and so the system is unstable; at least one root of the characteristic equation has a positive real part. We could of course have directly concluded instability because the coefficient a_3 of the characteristic polynomial is negative; however, the Routh table above gives more information regarding the placement of the roots in the complex plane, namely, that *two* roots of the characteristic polynomial have positive real parts. We then have the following result. \square

Result 8: The undamped gyroscopically stabilized two-degree-of-freedom (nongeneric) system given in Eq. (9) [Eq. (8)] with $\text{Trace}(D) = 0$ ($\text{Trace}(\bar{D}) = 0$), and $0 < k < 1$ ($0 < \bar{k}_1 < \bar{k}_2$) is unstable. \square

Recall from Result 2 that when $k = 1$, and $\text{Trace}(D) > 0$, the nongeneric system is unstable. Furthermore, Results 6, 7, and 8 can be summarized by saying that when $\text{Trace}(D) = 0$, the damped gyro-stabilized system cannot be made stable unless $k = 1$ ($\bar{k}_1 = \bar{k}_2$). When $k = 1$, in order for the damped gyro-stabilized

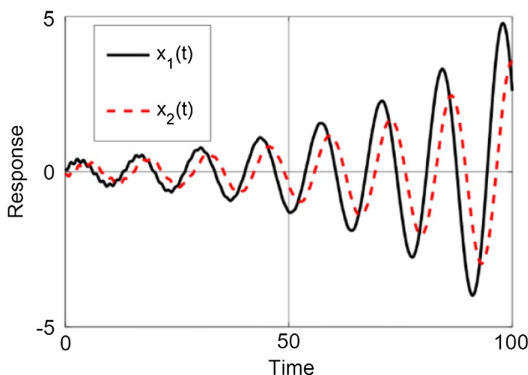


Fig. 8 Unstable response of system with $\bar{\alpha} = \bar{d} = 0.5$, $\bar{k}_1 = 2$, $\bar{k}_2 = 4$, $\bar{k}_3 = -2$, and $\bar{g} = 5$.

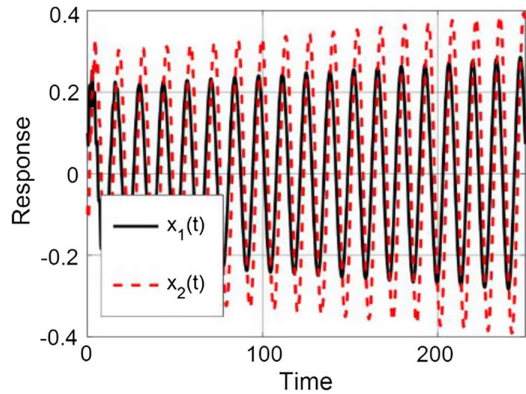


Fig. 9 Unstable response of system with $\bar{\alpha} = 0.2$, $\bar{d} = 0.4$, $\bar{k}_1 = 4$, $\bar{k}_2 = 2$, $\bar{k}_3 = -2$, $\bar{g} = 5$.

system (9) to be stable, we must have $\text{Trace}(D) = 0$, that is, $d = \alpha$. The damped system can then be made marginally stable if the conditions in Eqs. (58) and (59) are satisfied.

Numerical Example 5: We consider the system

$$\bar{\alpha} = \bar{d} = 0.5, \quad \bar{k}_1 = 2, \quad \bar{k}_2 = 4, \quad \bar{k}_3 = -2, \quad \bar{g} = 5 \quad (60)$$

Since $\bar{a}_1 = 0$ and $0 < \bar{k}_1 < \bar{k}_2$, the coefficient \bar{a}_3 of the characteristic equation is negative, and therefore the system is unstable. But more can be said by using the Routh table above.

Since $\bar{a}_2 + 1 = \bar{g}^2 - \bar{k}_1 - \bar{k}_2 - \bar{d}^2 + 1 = 19.75 > 0$, and $\bar{a}_4 = 4 > 0$, the Routh table shows that we must have 2 roots of the characteristic equation with positive real parts. The unstable response of the system is shown in Fig. 8, and the two roots with positive real parts are $0.0273 \pm 0.464i$, as expected.

Remark 13: Lastly, we consider the special, nongeneric case when $a_1, a_2, a_4 > 0$ and $a_3 = 0$ in Eq. (9). The condition $a_3 = 0$ implies that $d/\alpha = k$; that is, the ratio of the magnitudes of the dissipative damping and the positive velocity feedback equals the ratio of the stiffnesses \bar{k}_1 and \bar{k}_2 . The Routh table shows that there are two roots of the characteristic equation with positive real parts, and the system is unstable. \square

Result 9: The undamped gyroscopically stabilized two-degree-of-freedom (nongeneric) system given in Eq. (9) [Eq. (8)] with $\text{Trace}(D) > 0$ ($\text{Trace}(\bar{D}) > 0$) and $d/\alpha = \bar{k}_1/\bar{k}_2 = k$ ($d/\bar{\alpha} = \bar{k}_1/\bar{k}_2$), is unstable. \square

Numerical Example 6: Consider the system

$$\bar{\alpha} = 0.2, \quad \bar{d} = 0.4, \quad \bar{k}_1 = 4, \quad \bar{k}_2 = 2, \quad \bar{k}_3 = -2, \quad \bar{g} = 5$$

We note that the conditions given in Result 9 are satisfied. Figure 9 shows the unstable response of the system. The two roots of the characteristic equation with positive real parts are $0.0011 \pm 0.462i$.

Remark 14: The systems considered in Results 6–9 are nongeneric. In Results 4 and 5, it is shown that for every two-degree-of-freedom unstable potential matrix \bar{K}_1 , which can be gyro-stabilized through the use of a gyroscopic matrix \bar{G} , there exist numerous indefinite damping matrices \bar{D} that will render the system in Eq. (8) exponentially stable. The only exception is the nongeneric case when $\bar{k}_1 = \bar{k}_2$ in matrix \bar{K}_1 . And in that case Result 7 applies; only marginal stability can be achieved, and that too after additional restrictions are placed on the matrix \bar{G} . \square

III. Conclusions

An unstable potential system that has been gyroscopically stabilized becomes unstable in the presence of dissipative damping.

This well-established result has been handed down to the scientific and engineering communities since at least the last 100 years. Though nonintuitive, it is of considerable practical value since it correctly predicts the behavior of engineered systems as well those arising in nature. Today it is referred to in stability theory as the celebrated Kelvin–Tait–Chetaev theorem. The central purpose of this paper is to show that this long-standing paradigm—related to the work initiated by Tait in 1861 and completed with Kelvin in 1867—does not apply when the damping in a system is not purely dissipative. In this sense, the paper represents a paradigm shift in the theory of linear stability that says that gyro-stabilized unstable potential systems *can* be stabilized by the addition of linear damping.

To substantiate this paradigm shift, two-degree-of-freedom gyroscopically stabilized unstable potential systems are considered in this paper. Not all unstable potential systems with an even number of degrees of freedom can be gyroscopically stabilized. Restrictions need to be placed on both the unstable potential system (i.e., on the potential or stiffness matrix) and on the gyroscopic forces (gyroscopic matrix) employed to achieve such stabilization. Conditions are obtained under which a two-degree-of-freedom system with an unstable potential matrix can be stabilized through the use of gyroscopic forces. It is then shown that, excluding nongeneric systems, such gyroscopically stabilized systems can *always* be made *exponentially stable* through the addition of appropriate indefinite damping matrices.

This result conceptually points to a new methodology to control unstable potential systems that are gyroscopically stabilized. One way of physically generating an indefinite damping matrix is through the *simultaneous* provision of both *positive and negative* velocity feedback to different degrees of freedom of the undamped gyroscopically stabilized system. The cooperative interaction of these feedbacks engenders a somewhat surprising and nonintuitive type of control methodology that guarantees the capability of making the (generic) damped gyro-stabilized unstable potential system exponentially stable.

For two-degree-of-freedom systems a simple way of ascertaining which one of the two degrees of freedom receives negative velocity feedback (or dissipative damping) and which one receives positive velocity feedback is established. It is analytically and geometrically shown that a connected region exists in the “space” of indefinite damping matrices for which the damped gyroscopically stabilized system is guaranteed to be exponentially stable. This region of exponential stability is analytically delineated. The paper further proves that *every* (generic) two-degree-of-freedom system that can be gyroscopically stabilized can also be made exponentially stable by the use of an uncountably infinite number of indefinite damping matrices. Contrary to intuition, when the damping matrix is indefinite, progressively increasing the dissipative damping (for a given amount of positive velocity feedback), does not, in general, necessarily increase the likelihood of making a damped gyroscopically stabilized system stable, and can, in fact, make the system unstable. Instead, it is shown that for each value of the positive velocity feedback for which the damped system could be made exponentially stable, stability is assured only over a range, or interval, of values of the dissipative damping. Moreover, it is shown that in certain regimes (ranges) of positive velocity feedback no amount of dissipative damping can make the damped system exponentially stable. Conversely, there are regimes (ranges) of dissipative damping for which no amount of positive velocity feedback can make the system exponentially stable. Nonetheless, it is always possible to bestow exponential stability on the damped gyro-stabilized system.

The use of indefinite damping matrices whose traces are positive, and so imply that phase volumes exponentially shrink, appears to provide a practical way to achieve exponentially stable behavior of a damped gyro-stabilized (unstable) potential system. Special nongeneric situations (e.g., when the trace of the indefinite damping matrix is exactly zero) that lead only to marginal stability and even instability are considered in some detail.

The focus of this paper is to develop new concepts and ideas that have gone unaddressed thus far in the development of the theory of

linear stability—an area of significant interest to physicists, engineers, and mathematicians, because of its wide-spread practical utility. The central concept proposed herein is the following: in contradistinction to the well-known Kelvin–Tait–Chetaev paradigm, which says that instability is engendered through the use of positive definite damping matrices, this paper shows that gyro-stabilized unstable potential systems *can* be made *stable*, even exponentially so, through the use of appropriate indefinite damping matrices. To substantiate this new paradigm, a detailed investigation of two-degree-of-freedom gyro-stabilized systems is undertaken. Though two-degree-of-freedom systems can and do arise in engineering applications and in nature (e.g., astrodynamics; see Numerical Example 2), they are not commonly found in engineered systems, except when strong symmetries exist or when some modeling simplifications are applicable.

From a conceptual standpoint, this paper 1) provides a framework for making gyroscopically stabilized systems exponentially stable through the use of linear damping, thereby calling for a modification of the Kelvin–Tait–Chetaev paradigm; 2) demonstrates a practical control methodology to achieve this; and, most importantly, 3) points to new directions and paradigms in our understanding of linear stability of dynamical systems.

Appendix: Conditions for Gyroscopic Stabilization of Unstable Potential Systems

Result: In the notation given in Eq. (8), a two-degree-of-freedom, unstable potential system can be gyroscopically stabilized through the use of the two by two matrix \bar{G} , if and only if

$$(i) \bar{K}_1 > 0, \quad \text{with } \bar{k}_1, \bar{k}_2 > 0, \quad \text{and} \quad (A1)$$

$$(ii) \bar{g}^2 = \bar{k}_1 + \bar{k}_2 + 2\sqrt{\text{Det}(\bar{K}_1)} + \bar{\delta}^2, \quad \text{for some } \bar{\delta} \neq 0 \quad (A2)$$

Proof: Using the notation shown in Eq. (8), consider the unstable potential system

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\ddot{x}} - \underbrace{\begin{bmatrix} \bar{k}_1 & \bar{k}_3 \\ \bar{k}_3 & \bar{k}_2 \end{bmatrix}}_{\bar{K}_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0, \quad \bar{K}_1 \neq 0 \quad (A3)$$

that is stabilized by the addition of a gyroscopic matrix \bar{G} to yield the undamped gyroscopically stabilized system

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{bmatrix} 0 & \bar{g} \\ -\bar{g} & 0 \end{bmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} \bar{k}_1 & \bar{k}_3 \\ \bar{k}_3 & \bar{k}_2 \end{bmatrix}}_{\bar{K}_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = 0 \quad (A4)$$

The characteristic polynomial of the system described in Eq. (A4) is

$$z(\lambda) := \lambda^4 + (\bar{g}^2 - \bar{k}_1 - \bar{k}_2)\lambda^2 + \text{Det}(\bar{K}_1) = 0 \quad (A5)$$

where $\text{Det}(\bar{K}_1) = \bar{k}_1\bar{k}_2 - \bar{k}_3^2$.

For stability of the system described in Eq. (A4), we must then have $\text{Det}(\bar{K}_1) > 0$ and $(\bar{g}^2 - \bar{k}_1 - \bar{k}_2) > 0$. Furthermore, considering the polynomial as a quadratic in $\mu := \lambda^2$, its roots are given by

$$2\mu_{1,2} = -(\bar{g}^2 - \bar{k}_1 - \bar{k}_2) \pm \sqrt{(\bar{g}^2 - \bar{k}_1 - \bar{k}_2)^2 - 4\text{Det}(\bar{K}_1)} \quad (A6)$$

For the system described by Eq. (A4) to be stable, $\mu_{1,2}$ must be real and negative. This makes the four roots of the biquadratic in λ purely imaginary, and stability is assured when these roots are distinct. Hence, we require

$$(\bar{g}^2 - \bar{k}_1 - \bar{k}_2)^2 > 2\text{Det}(\bar{K}_1) \quad (\text{A7})$$

or,

$$\bar{g}^2 > \bar{k}_1 + \bar{k}_2 + 2\sqrt{\text{Det}(\bar{K}_1)}$$

Thus, for the unstable two-degree-of-freedom potential system described by Eq. (A3) to be gyroscopically stabilized it is required that

$$(i) \text{Det}(\bar{K}_1) > 0, \text{ and} \quad (\text{A8})$$

$$(ii) \bar{g}^2 = \bar{k}_1 + \bar{k}_2 + 2\sqrt{\text{Det}(\bar{K}_1)} + \bar{\delta}^2, \text{ for some } \bar{\delta} \neq 0 \quad (\text{A9})$$

Furthermore, since the potential system described by Eq. (A3) is unstable, the matrix \bar{K}_1 must have at least one positive eigenvalue, or one zero eigenvalue.

Consider first the case when \bar{K}_1 has one positive eigenvalue. Then from Eq. (A8), for the system described by Eq. (A4) to be stable we require, $\text{Det}(\bar{K}_1) > 0$. But since $\text{Det}(\bar{K}_1)$ is the product of the eigenvalues of \bar{K}_1 , its other eigenvalue must also be positive, so that $\bar{K}_1 > 0$. This then implies that $\bar{k}_1, \bar{k}_2 > 0$ in Eq. (A4), and Eq. (A3).

Consider next the case when \bar{K}_1 has one zero eigenvalue. Then its other eigenvalue cannot also be zero, since then \bar{K}_1 would have to be the zero matrix, which we assume is not the case. Hence only one eigenvalue of \bar{K}_1 can be zero. But if that is true, then $\text{Det}(\bar{K}_1) = 0$, and the characteristic polynomial $z(\lambda)$ of the system described by Eq. (A4) reduces to

$$\lambda^2[\lambda^2 + (\bar{g}^2 - \bar{k}_1 - \bar{k}_2)] = 0 \quad (\text{A10})$$

with a double root at $\lambda = 0$. Thus the system described by Eq. (A4) is unstable and has a polynomial-type instability no matter what value of \bar{g} is chosen. Hence, no gyroscopic stabilization is possible of an unstable potential system in which the matrix \bar{K}_1 has one zero eigenvalue. Hence, for the unstable potential system to be gyroscopically stabilized, we require that

$$(i) \bar{K}_1 > 0, \text{ and}$$

$$(ii) \bar{g}^2 = \bar{k}_1 + \bar{k}_2 + 2\sqrt{\text{Det}(\bar{K}_1)} + \bar{\delta}^2, \text{ for some } \bar{\delta} \neq 0 \quad (\text{A11})$$

Note that in Eq. (A4), an interchange of \bar{k}_1 and \bar{k}_2 in matrix \bar{K}_1 leaves the conditions given in Eq. (A11) unchanged.

The “only if” part of the result follows directly by using the two conditions [Eqs. (A1) and (A2)] in equation Eq. (A6).

Finally, when $\bar{k}_1 = k, \bar{k}_2 = 1$, and $\bar{g} = g$ as in Eq. (9), in which the dots refer to differentiation with respect to the scaled time τ , the second of these relations gives

$$g^2 = 1 + k + 2\sqrt{\text{Det}(K_1)} + \delta^2, \text{ for some } \delta \neq 0 \quad (\text{A12})$$

This completes the proof. \square

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